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A Density Perturbation Method to Study the Eigenstructure of Two-Phase Flow Equation Systems

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Many interesting and challenging physical mechanisms are concerned with the mathematical notion of eigenstructure. In two-fluid models, complex phasic interactions yield a complex eigenstructure which may raise numerous problems in numerical simulations. In this paper, we develop a perturbation method to examine the eigenvalues and eigenvectors of two-fluid models. This original method, based on the stiffness of the density ratio, provides a convenient tool to study the relevance of pressure momentum interactions and allows us to get precise approximations of the whole flow eigendecomposition for minor requirements. Roe scheme is successfully implemented and some numerical tests are presented.

1. INTRODUCTION

The numerical simulation of two-phase flow is a thrilling mathematical and industrial research subject and is of prime interest for the safety studies of nuclear reactors or for the flow encountered in steam generators of pressurized water reactors (PWR). The complete local description of such flows is very complex and cannot be used for industrial simulation. An averaging technique is used to derive models suitable for computation [7, 16]. When motions of the two-phases are strongly coupled, the relative velocity of the two-fluid is small and a simplified model is obtained; it consists of a system of conservation laws for the mass, momentum, and energy balance equations for the gas–liquid mixture. However, such mixture models are inefficient when the kinematic disequilibrium becomes significant, which typically occurs, for instance, in an annular flow or in the U-bend of a PWR's steam generator. In this circumstances, a set of equations for each phase must be considered and a two-fluid model must be simulated.

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Donor-cell differencing methods, using staggered grids, have been widely used for two-phase flow computations [22, 28]. In general they introduce a large amount of numerical diffusion. Moreover, they are not well suited for complex geometries. An alternative is to use finite volumes or finite elements. Finite element methods have been successfully implemented for the simulation of steady states of mixture models [12]. Their extension to transient flow is in progress. Also, a widespread use of upwind schemes [4, 9–11, 13, 15, 21, 30] is growing in the simulation of fast transient flows. These schemes, introduced in the 1960s for single-phase flows, are very convenient to predict gas–liquid mixture evolution but their two-fluid extension is not straightforward and has been a subject of intensive research for the last 10 years [3, 5, 18, 27, 29, 32–35].

If U denotes the vector of unknowns, the constitutive equations for the averaged two-fluid flow model are given by

$$\partial_t U + \partial_x f(U) + G(U)\partial_x U + \partial_x D(U, \partial_x U) = S(U),$$

which means that the evolution of U is governed by convection, diffusion, and source terms. Our aim in this paper is to compute finite volume schemes for hyperbolic systems [9, 10, 13, 15]. This requires a good understanding of the convection part of the equation. Thus, in this work, we do not take into account the source and diffusion terms. We will study a system of first-order partial differential equations of the type:

$$\partial_t U + \partial_x f(U) + G(U)\partial_x U = 0.$$

In order to solve this basic system, several difficulties have to be overcome:

- First, the system is not in conservation law form, due to the term $G(U)\partial_x U$, and must be carefully handled in the presence of discontinuities [6, 18]. Here, we do not discuss this issue and follow the approach of [18].
- Closure relations are needed because the system contains more unknowns than equations. These relations are derived from empirical data, which is a major source of error [7, 33]. In this paper, for simplicity, the liquid will be assumed to be incompressible and we will consider perfect gas for the vapor phase.
- Upwinding requires some knowledge of the eigenstructure of $\partial_U f(U) + G(U)$, where $\partial_U f$ denotes the jacobian of f. Here, contrary to the case of single-phase flows, complex phasic interactive processes, such as pressure interaction, complicate the computation of the eigenvalues and eigenvectors of the system.
- Finally, the system might fail to be hyperbolic. This might lead to an ill-posed problem with oscillations in finite volume simulations ([25]).

In this work, we give particular attention to those two last difficulties. Although an easy computation of the eigenstructure is of prime interest in industrial applications, few approaches have been proposed to solve this problem. Numerical algorithms can be used. However, the eigenelements are needed at each face of the mesh and, even if improvements are in progress, we think that the use of a numerical algorithm may be expensive in CPU time. Here, we prefer another approach in which simplifications in the computations are obtained thanks to physical approximations; in [29], the case of small void fraction is studied, whereas in [18], explicit calculations are done, assuming that the relative velocity is very small compared to the average speed of sound. Also, some authors [27] suggest

taking into account a two-pressure model, which obviously simplifies the computations but requires an additional closure relation. Related to this difficulty, the hyperbolicity of two-fluid models is not clearly understood; if we want to work with a hyperbolic model, we have to modify the system by adding some differential correction terms, presumably neglected during the averaging process. The interfacial pressure correction proposed in [35] is one such example.

In this paper, we remark that such a complicated system is simplified when we neglect the pressure gradient in the liquid phase. Because the vapor density is smaller than the liquid density, we introduce, by means of a scaling, the ratio of the phasic densities which enables us to write the system as a simple system, with no pressure gradient in the liquid phase, plus a perturbation which contains this pressure gradient. Then we can use mathematical perturbation techniques to derive approximations of eigenvectors and eigenvalues of the whole system [2, 14, 17, 23]. We show that our density perturbation method is an efficient tool to study the relevance of the differential correction terms and consequently provides a convenient frame to study the hyperbolicity of such models and to compute upwind schemes for minor requirements.

The paper is organized as follows. In Section 2, we present the two-fluid model. The density perturbation method is introduced in Section 3, as well as some mathematical background. Then in Section 4, we get results on the hyperbolicity of the system with respect to pressure corrections. We derive approximate expressions of the eigenvalues and eigenvectors in Section 5. We apply the method in the context of the Roe scheme for two-fluid computations in Section 6, and present numerical results in the last section.

2. THE TWO-FLUID MODEL

The mathematical evolution of the flow is governed by a physical system of balance equations. It consists in two phasic mass equations and two phasic momentum equations. As in [18], we do not deal with the two phasic energy equations. The analysis of the system performed in our work can be easily extended to the full system containing these equations.

The subscripts g and l refer to the gas and the liquid phase, respectively. The nomenclature is as follows: α is the volume fraction ($\alpha_g + \alpha_l = 1$), ρ is the density, v is the velocity, and p is the pressure (see Delhaye *et al.* [7] or Ishii [16] for a complete discussion).

Let U be the unknown vector of conservative variables:

$$U = \begin{pmatrix} \alpha_l \rho_l \\ \alpha_g \rho_g \\ \alpha_l \rho_l v_l \\ \alpha_g \rho_g v_g \end{pmatrix}. \tag{2.1}$$

The convection part of the governing equations is given below:

- *mass conservation equation*,

$$\partial_t(\alpha_k \rho_k) + \partial_x(\alpha_k \rho_k v_k) = 0, \quad k = l, g; \tag{2.2}$$

- *momentum conservation equation*,

$$\partial_t(\alpha_k \rho_k v_k) + \partial_x\left(\alpha_k \rho_k v_k^2\right) + \alpha_k \partial_x p_k + \left(p_k - p_k^i\right)\partial_x \alpha_k = 0, \quad k = l, g. \tag{2.3}$$

The system contains more unknowns than equations and we need closure relations:

• *Constitutive laws.*

—For simplicity, throughout this paper, the liquid phase is assumed to be incompressible:

$$\rho_l = \text{const}; \tag{2.4}$$

—the gas pressure is governed by a perfect gas law:

$$p_g = p = p\left(\frac{U_2}{1 - U_1/\rho_l}\right) = \Gamma \rho_g^{\gamma}; \tag{2.5}$$

—and the liquid pressure is assumed to be equal to the gas pressure:

$$p_l = p. \tag{2.6}$$

• *Interface pressure corrections.* p_k^i is the phasic pressure at the gas–liquid interface, presumably neglected during the averaging process. The differential terms $(p - p_k^i)\partial_x \alpha_k$, called pressure corrections, must be modelled in order to close the system. Several pressure corrections can be found in the literature:

—The simplest one is the *common pressure* relation, used, for example, in the RELAP5 differential model (actually a virtual mass term, included in the RELAP5 model, has been omitted here for the sake of simplicity; see [24, 28]):

$$\begin{vmatrix} p - p_g^i = 0 \\ p - p_l^i = 0. \end{vmatrix} \tag{2.7}$$

—Detailed closure models for bubbly flows have been presented by Lahey in [19]:

$$\begin{vmatrix} p - p_g^i = 0 \\ p - p_l^i = C_p(\alpha_g)\rho_l (v_g - v_l)^2. \end{vmatrix} \tag{2.8}$$

—Whereas some authors take into account pressure corrections for all flow configurations in such a way that we always have a hyperbolic system, I. Toumi ([35]) suggests

$$\begin{vmatrix} p - p_g^i = p - p_l^i \\ p - p_l^i = \alpha_g \delta \rho_l (v_g - v_l)^2 \end{vmatrix} \tag{2.9}$$

with $\delta \simeq \frac{1}{2}$.

—Bestion proposes another development for this term in the CATHARE code model [1]:

$$\begin{vmatrix} p - p_g^i = p - p_l^i \\ p - p_l^i = \dfrac{\alpha_g \alpha_l \rho_g \rho_l}{\alpha_g \rho_l + \alpha_l \rho_g} (v_g - v_l)^2. \end{vmatrix} \tag{2.10}$$

• In fact, noticing that $\alpha_g = 1 - U_1/\rho_l$, all these pressure corrections can be written in the generalized form

$$\left(p - p_k^i\right)\partial_x \alpha_k = \theta_k(U)\partial_x U_1 \tag{2.11}$$

and satisfy some natural requirements:

—We see that $p - p_k^i$ is homogeneous to a friction:

$$\left[\left[p - p_k^i\right]\right] = \frac{kg}{m \cdot s^2}. \tag{2.12}$$

—Pressure corrections must vanish when the gas velocity is equal to the liquid velocity:

$$\left(p - p_k^i\right) \to 0 \quad \text{as} \quad (v_g - v_l) \to 0. \tag{2.13}$$

—In order to get a conservation law for the momentum of the mixture of the fluid, we might ask these corrections to satisfy the requirement:

$$p - p_g^i = p - p_l^i. \tag{2.14}$$

But actually, we can see that the correction (2.8) does not satisfies this requirement so we do not insist on that point.

Finally, we have a closed system,

$$\partial_t U + \partial_x f(U) + G(U)\partial_x U = 0 \tag{2.15}$$

with

$$f(U) = \begin{pmatrix} \alpha_l \rho_l v_l \\ \alpha_g \rho_g v_g \\ \alpha_l \rho_l v_l^2 \\ \alpha_g \rho_g v_g^2 \end{pmatrix} \tag{2.16}$$

and

$$G(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_l \partial_{U_1} p + \theta_l(U) & \alpha_l \partial_{U_2} p & 0 & 0 \\ \alpha_g \partial_{U_1} p + \theta_g(U) & \alpha_g \partial_{U_2} p & 0 & 0 \end{pmatrix}. \tag{2.17}$$

The mathematical modeling of $\theta_g(U)$ and $\theta_l(U)$ depends on the pressure correction chosen. Moreover, due to term (2.17), the two-fluid system (2.15) cannot be put in a conservation law form.

We want to point out that the density perturbation method (D.P.M) aims at studying the influence of pressure corrections on the eigenstructure of two-fluid models without debating their physical relevance, which may be controversial [24, 25].

3. THE DENSITY PERTURBATION METHOD

3.1. *Scaling of the Densities*

We introduce two characteristic densities ρ_g^0 and ρ_l^0 and the new variables $\tilde{\rho}_g$, $\tilde{\rho}_l$, $\tilde{U}$:

$$\tilde{\rho}_g = \rho_g / \rho_g^0; \; \tilde{\rho}_l = \rho_l / \rho_l^0, \tag{3.18}$$

$$\tilde{U} = \begin{pmatrix} \alpha_l \tilde{\rho}_l \\ \alpha_g \tilde{\rho}_g \\ \alpha_l \tilde{\rho}_l v_l \\ \alpha_g \tilde{\rho}_g v_g \end{pmatrix}. \tag{3.19}$$

We set

$$\tilde{p}(\tilde{\rho}_g) = \frac{1}{\rho_g^o}\, p(\rho_g) \tag{3.20}$$

and we define

$$\epsilon = \rho_g^0 / \rho_l^0. \tag{3.21}$$

A dimension analysis of (2.11) and (2.12) proves that

$$[[\theta_k(U)]] = \frac{m^2}{s^2}, \tag{3.22}$$

Thus, θ_k is not proportional to a density and we can write

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \theta_l(U) & 0 & 0 & 0 \\ \theta_g(U) & 0 & 0 & 0 \end{pmatrix} = \sum_{q=q_0}^{q_1} \epsilon^q B_q(\tilde{U}). \tag{3.23}$$

In the examples cited above and treated below, we will see that $q_0 = -1$ or 0, and $q_1 = 1$. Then, we get

$$\partial_t \tilde{U} + \partial_x f(\tilde{U}) + H(\tilde{U})\partial_x \tilde{U} = 0, \tag{3.24}$$

with

$$H(\tilde{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon\alpha_l \partial_{\tilde{U}_1} \tilde{p} & \epsilon\alpha_l \partial_{\tilde{U}_2} \tilde{p} & 0 & 0 \\ \alpha_g \partial_{\tilde{U}_1} \tilde{p} & \alpha_g \partial_{\tilde{U}_2} \tilde{p} & 0 & 0 \end{pmatrix} + \sum_{q=q_0}^{q_1} \epsilon^q B_q(\tilde{U}). \tag{3.25}$$

Expressions (3.24) and (3.25) must be compared with expressions (2.15) to (2.17). The density ratio depends on the configuration of the flow. For instance,

- steam generator, $\epsilon \simeq 5 \times 10^{-2}$
- air–water bubbly flow $\epsilon \simeq 10^{-3}$.

Hence, according to (3.25)., ϵ can be seen as an efficient perturbation parameter. As soon as the vapor volume α_g is not stiff ($\alpha_g > \epsilon/(1+\epsilon)$), the system can be written as a simpler one, perturbed by small terms.

3.2. *Separation of the Phasic Pressure Interactions*

From now on, we will omit ˜ in the notations. We write system (3.24) in the form

$$\partial_t U + A(U)\partial_x U = 0, \tag{3.26}$$

with

$$A(U) = \partial_U f(U) + H(U). \tag{3.27}$$

The small parameter ϵ introduced above defines a splitting of the matrix $A(U)$,

$$A(U) = \frac{1}{\epsilon} A_{-1}(U) + A_0(U) + \epsilon A_1(U) \tag{3.28}$$

with

$$A_{-1} = B_{-1}(U) \tag{3.29}$$

(actually, we will see that for all pressure corrections, except the (2.9) one, $A_{-1} = 0$):

$$A_0(U) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_l^2 & 0 & 2v_l & 0 \\ \alpha_g \partial_{U_1} p & \alpha_g \partial_{U_2} p - v_g^2 & 0 & 2v_g \end{pmatrix} + B_0(U) \tag{3.30}$$

and

$$A_1(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_l \partial_{U_1} p & \alpha_l \partial_{U_2} p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + B_1(U). \tag{3.31}$$

We have split the phasic pressure interactions: the matrix $(1/\epsilon)A_{-1}(U) + A_0(U)$ would represent a system with no pressure gradient on the liquid phase and it is expected that its eigenvalues and eigenvectors are easy to find. Hence, perturbation methods are a natural way to take into account the contribution of $\epsilon A_1(U)$ in order to study the eigenstructure of the original matrix $A(U)$.

3.3. *Perturbation Analysis, Definitions, and Results*

Perturbation theory for linear operators [14, 17] provides a convenient way to analyse the well-posed nature of the system and to obtain computable approximations of the original matrix for minor requirements. We now go into our proper subject and review the necessary mathematical background (see the references for a more comprehensive treatment).

Let B be a real matrix of dimension n (in our applications, $n = 4$). We define σ_B, the set of eigenvalues:

$$\sigma_B = \{\lambda \text{ complex}; \det(B - \lambda Id) = 0\} = \{\lambda_1, \ldots, \lambda_k\}, \tag{3.32}$$

where "det" denotes the determinant which can be written in the form

$$\det(B - \lambda Id) = (\lambda_0 - \lambda)^{m_0} (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}; \tag{3.33}$$

m_j is the algebraic multiplicity of the eigenvalue λ_j and

$$\sum_{j=1}^{k} m_j = n. \tag{3.34}$$

Moreover, at each eigenvalue there is associated the eigen-subspace K_j:

$$K_j = \{x / Bx = \lambda_j x\} = \mathrm{Ker}(B - \lambda_j Id) \tag{3.35}$$

and g_j, the dimension of K_j, is the geometric multiplicity of the eigenvalue λ_j:

$$g_j = \dim K_j. \tag{3.36}$$

B is called the unpertubed operator and we assume that its eigenvalues and eigenvectors are easy to find. Let H be a real matrix; we want to study of the evolution of the eigenstructure from B to B':

$$B' = B + \epsilon H. \tag{3.37}$$

The perturbation theory of linear operators is an intense research subject. Here, we use the following results that can be found in [2] and are of prime interest in two-fluid applications.

Let λ_j be an eigenvalue of B; there are m_j eigenvalues λ'_{j_i} (of B') obtained from the perturbation of the eigenvalue λ_j and it can be proved that

$$\min\left\{ \left|\lambda_j - \lambda'_{j_i}\right|; i = 1 \dots m_j \right\} = O(\epsilon^{g_j/m_j}). \tag{3.38}$$

This estimate will be of prime interest in Section 5. Actually, the analysis of the pressure corrections (2.7), (2.8), (2.9), and (2.10) for two-fluid models will lead to two crucial distinct cases:

- *The strictly diagonalizable case.* The unperturbed operator B is diagonalizable with distinct real eigenvalues. Hence, $m_j = g_j = 1$ for all j and it can be proved [2] that, for small ϵ, B' is diagonalizable with distinct real eigenvalues.
- *The degenerate case.* σ_B is real but the unperturbed operator B is no longer diagonalizable; it has a nontrivial 2×2 Jordan block. The evolution of the degenerate root is not straightforward and we will see that B' may be diagonalizable with distinct complex eigenvalues even for arbitrary small ϵ.

We will say that the system is well-posed, or hyperbolic, when B' is diagonalizable with only real eigenvalues.

Remark 3.1. In all the applications, except for the pressure correction (2.9), we will set $B = A_0$ and $H = A_1$. When studying the pressure correction (2.9), we will have a matrix of the type $(1/\epsilon)C + B + \epsilon H$. The above analysis can easily be extended to this case but will not be fully developed in this paper.

4. HYPERBOLICITY

4.1. *Strictly Diagonalizable Systems*

4.1.1. Study of the pressure correction (2.8). We require $C_p > 0$ for $0 < \alpha_g < 1$. The application of the D.P.M. leads to the following expression of A_{-1}, A_1, and A_0:

$$A_{-1} = 0, \tag{4.39}$$

$$A_1 = \alpha_l \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_{U_1} p & \partial_{U_2} p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.40}$$

and

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_l^2 + C_p(\alpha_g)(v_g - v_l)^2 & 0 & 2v_l & 0 \\ \alpha_g \partial_{U_1} p & \alpha_g \partial_{U_2} p - v_g^2 & 0 & 2v_g \end{pmatrix}. \tag{4.41}$$

Using the notations

$$\alpha_g \partial_{U_2} p = c_m^2, \quad v_r = v_g - v_l, \tag{4.42}$$

we can find eigenvalues of A_0 as trivial roots of P_0,

$$P_0(\lambda) = \left[(\lambda - v_l)^2 - C_p(\alpha_g) v_r^2\right]\left[(\lambda - v_g)^2 - c_m^2\right], \tag{4.43}$$

and, as soon as $v_r \neq 0$, we easily get the decomposition of A_0 (the equal velocity case will be treated in Subsection 4.2):

$$A_0 = P D P^{-1}, \tag{4.44}$$

with

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = \begin{pmatrix} v_g - c_m & 0 & 0 & 0 \\ 0 & v_l - \sqrt{C_p v_r^2} & 0 & 0 \\ 0 & 0 & v_l + \sqrt{C_p v_r^2} & 0 \\ 0 & 0 & 0 & v_g + c_m \end{pmatrix}, \tag{4.45}$$

$$P = \begin{pmatrix} 0 & c_m^2 - v_r^2\left(1 + \frac{|v_r|}{v_r}\sqrt{C_p}\right)^2 & c_m^2 - v_r^2\left(1 - \frac{|v_r|}{v_r}\sqrt{C_p}\right)^2 & 0 \\ 1 & -\frac{\rho_g}{\rho_l} c_m^2 & -\frac{\rho_g}{\rho_l} c_m^2 & 1 \\ 0 & \left(c_m^2 - v_r^2\left(1 + \frac{|v_r|}{v_r}\sqrt{C_p}\right)^2\right)\lambda_2 & \left(c_m^2 - v_r^2\left(1 - \frac{|v_r|}{v_r}\sqrt{C_p}\right)^2\right)\lambda_3 & 0 \\ \lambda_1 & -\frac{\rho_g}{\rho_l} c_m^2 \lambda_2 & -\frac{\rho_g}{\rho_l} c_m^2 \lambda_3 & \lambda_4 \end{pmatrix} \tag{4.46}$$

(remember that densities are now dimensionless). A_0 is strictly diagonalizable and, according to Subsection 3.3, it is expected that the small perturbation ϵA_1 will yield a well-posed two-fluid model.

4.1.2. Study of the pressure correction (2.9). The study of this correction is similar to the previous one. The application of the D.P.M. leads to the same expression of A_1, whereas A_0 and A_{-1} are now given by

$$A_{-1}(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_g \delta (v_g - v_l)^2 & 0 & 0 & 0 \end{pmatrix}, \tag{4.47}$$

$$A_0(U) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_l^2 + \alpha_g \delta (v_g - v_l)^2 & 0 & 2v_l & 0 \\ \alpha_g \partial_{U_1} p & \alpha_g \partial_{U_2} p - v_g^2 & 0 & 2v_g \end{pmatrix}. \tag{4.48}$$

Then, setting

$$A_{-1,0} = \frac{1}{\epsilon} A_{-1} + A_0, \tag{4.49}$$

we easily get the eigendecomposition of $A_{-1,0}$:

$$A_{-1,0} = PDP^{-1}, \tag{4.50}$$

with

$$D = \begin{pmatrix} v_g - c_m & 0 & 0 & 0 \\ 0 & v_l - \sqrt{\alpha_g \delta v_r^2} & 0 & 0 \\ 0 & 0 & v_l + \sqrt{\alpha_g \delta v_r^2} & 0 \\ 0 & 0 & 0 & v_g + c_m \end{pmatrix}, \tag{4.51}$$

$$P = \begin{pmatrix} 0 & c_m^2 - v_r^2 \big(1 + \frac{|v_r|}{v_r} \sqrt{\alpha_g \delta}\big)^2 & c_m^2 - v_r^2 \big(1 - \frac{|v_r|}{v_r} \sqrt{\alpha_g \delta}\big)^2 & 0 \\ 1 & -\frac{\rho_g}{\rho_l} c_m^2 + \frac{\alpha_g \delta}{\epsilon} v_r^2 & -\frac{\rho_g}{\rho_l} c_m^2 + \frac{\alpha_g \delta}{\epsilon} v_r^2 & 1 \\ 0 & c_m^2 - v_r^2 \big(1 + \frac{|v_r|}{v_r} \sqrt{\alpha_g \delta}\big)^2 \lambda_2 & c_m^2 - v_r^2 \big(1 - \frac{|v_r|}{v_r} \sqrt{\alpha_g \delta}\big)^2 \lambda_3 & 0 \\ \lambda_1 & \big(-\frac{\rho_g}{\rho_l} c_m^2 + \frac{\alpha_g \delta}{\epsilon} v_r^2\big) \lambda_2 & \big(-\frac{\rho_g}{\rho_l} c_m^2 + \frac{\alpha_g \delta}{\epsilon} v_r^2\big) \lambda_3 & \lambda_4 \end{pmatrix}. \tag{4.52}$$

We obtain an expression similar to (4.45), (4.46) with $C_p = \alpha_g \delta$, except for the presence of the term $(\alpha_g \delta / \epsilon) v_r^2$ only in the set of eigenvectors. Hence, the two-fluid system is again expected to be well-posed.

4.2. *A Degenerate System; The Common Pressure Model*

Let us study the hyperbolicity of the *common pressure* model (2.7). The computation leads to $A_{-1} = 0$, whereas $A_0(U)$ and $A_1(U)$ are now given by

$$A_0(U) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -v_l^2 & 0 & 2v_l & 0 \\ \alpha_g \partial_{U_1} p & \alpha_g \partial_{U_2} p - v_g^2 & 0 & 2v_g \end{pmatrix}, \tag{4.53}$$

$$A_1(U) = \alpha_l \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_{U_1} p & \partial_{U_2} p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.54}$$

Here, $A_0(U)$ is not diagonalizable but we get the Jordan decomposition

$$A_0 = R J R^{-1} \tag{4.55}$$

with

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = \begin{pmatrix} v_g - c_m & 0 & 0 & 0 \\ 0 & v_l & 1 & 0 \\ 0 & 0 & v_l & 0 \\ 0 & 0 & 0 & v_g + c_m \end{pmatrix} \tag{4.56}$$

and

$$R = \begin{pmatrix} 0 & \lambda_2\left(v_r^2 - c_m^2\right) & v_l^2 - v_g^2 + c_m^2 & 0 \\ 1 & \lambda_2 \frac{\rho_g}{\rho_l} c_m^2 & -\frac{\rho_g}{\rho_l} c_m^2 & 1 \\ 0 & \lambda_2^2\left(v_r^2 - c_m^2\right) & -2 v_r \lambda_2^2 & 0 \\ \lambda_1 & \lambda_2^2 \frac{\rho_g}{\rho_l} c_m^2 & 0 & \lambda_4 \end{pmatrix}. \tag{4.57}$$

Now, in order to study the hyperbolicity of the model, we must look at the perturbation operator in some detail. Lidskii [23] has proved that (with the same notation as in (3.38)):

$$\begin{cases} \lambda'_{2_1} - \lambda_2 = \sqrt{\epsilon}\,\kappa_1 + o(\sqrt{\epsilon}) \\ \lambda'_{2_2} - \lambda_2 = \sqrt{\epsilon}\,\kappa_2 + o(\sqrt{\epsilon}), \end{cases} \tag{4.58}$$

where κ_1 and κ_2 are the square roots of κ^2:

$$\kappa^2 = l_3 A_1 r_2 \tag{4.59}$$

(r_2 is the second column of R and l_3 the third line of R^{-1}). Here, we get the following value:

$$\kappa^2 = -\frac{\alpha_l \rho_g}{\alpha_g \rho_l} \frac{v_r^2}{1 - (v_r/c_m)^2}. \tag{4.60}$$

Subsonic multiphase flows are considered for which $v_r/c_m < 1$; it follows that λ'_{2_1} and λ'_{2_2} velocities are complex and $A_0 + \epsilon A_1$ is diagonalizable with distinct complex eigenvalues. The density perturbation method provides an original way to see that the common pressure model is ill-posed, which was already known by other methods.

Remark 4.2. It may also happen that the addition of some pressure corrections leads to real eigenvalues even if A_0 is only Jordanizable. In fact, such degenerate systems may be diagonalized with real eigenvalues under the condition $\kappa^2 > 0$. This is the case for some pressure corrections in the order of ϵ and, as an example, the study of the Cathare correction (2.10) is left to the reader.

Remark 4.3. In the equal-velocity case ($v_r = 0$), according to requirement (2.13), no pressure correction can be taken into account. Consequently, $A_{-1} = 0$ and A_1, A_0 are given by (4.53) and (4.54) with $v_r = 0$. In particular, $\kappa = 0$ and the system is not diagonalizable; such singular phase points have been studied differently in the case of the transition to detonation in reactive granular materials and are known as resonant points [8].

5. APPROXIMATE EIGENSTRUCTURE

We have studied the well-posed nature of two-fluid systems. Now we want to get approximate expressions of the eigenvalues and eigenvectors for minor requirements. The eigenstructure of B is explicitly known and we want to derive approximations of the eigenstructure of $B + \epsilon H$. To this end, several algorithms can be found in the literature [2, 14, 23]. Because each eigenvalue or eigenvector of $B + \epsilon H$ admits an expansion in fractional powers of ϵ, we only consider algorithms based on explicit formulas.

5.1. *Strictly Diagonalizable Systems*

We are searching for $x(\epsilon)$ and $\lambda(\epsilon)$ such that

$$(B + \epsilon H)x(\epsilon) = \lambda(\epsilon)x(\epsilon). \tag{5.61}$$

According to estimate (3.38), we expand these terms into a series about ϵ

$$\left| \begin{array}{l} x(\epsilon) = \displaystyle\sum_{i\geq 0} \epsilon^i x(i) \\ \lambda(\epsilon) = \displaystyle\sum_{i\geq 0} \epsilon^i \lambda(i) \end{array} \right. \tag{5.62}$$

and, substituting (5.62) in (5.61), we get

$$Bx(0) + \sum_{i\geq 1} \epsilon^i (Bx(i) + Hx(i-1)) = \lambda(0)x(0) + \sum_{i\geq 1} \epsilon^i \left(\sum_{j=0}^{i} x(j)\lambda(i-j) \right). \tag{5.63}$$

The formal identification of the series provides the approximations

$$Bx(0) = \lambda(0)x(0), \tag{5.64}$$

and for $i \geq 1$ (where we have set $x^l(0)B = \lambda(0)x^l(0)$)

$$\lambda(i) = x^l(0)Hx(i-1) - x^l(0) \sum_{j=1}^{i-1} x(j)\lambda(i-j), \tag{5.65}$$

$$(B - \lambda(0))x(i) = -Hx(i-1) + \sum_{j=0}^{i-1} x(j)\lambda(i-j). \tag{5.66}$$

One would expect that such formal series converge toward the eigenvalues and eigenvectors of the whole system. But the convergence is not obvious (see [2]). At least, it should be required that the condition number $\lambda(1) = x^l(0)Hx(0)$ be small. In the case of the model proposed by Lahey, studied in Section 4.1.1, setting $B = A_0$ and $H = A_1$, this term can be computed directly and we obtain for the eigenvalue λ_2

$$\lambda_2^{\mathrm{RTL}}(1) = \mu(C_p) \tag{5.67}$$

with

$$\mu(C_p) = \frac{1}{2\sqrt{C_p}} \frac{\alpha_l \rho_g}{\alpha_g \rho_l} \frac{v_r^2}{|v_r|} \frac{(1 + (v_r/|v_r|)\sqrt{C_p})^2}{1 - \left(v_r^2/c_m^2\right)(1 + (v_r/|v_r|)\sqrt{C_p})^2}. \tag{5.68}$$

Whereas, for the correction (2.9) proposed by Toumi, we obtain

$$\lambda_2^{IT}(1) = \mu(\alpha_g \delta) + \frac{1}{\epsilon} \nu(\alpha_g \delta) \tag{5.69}$$

with

$$\nu(\alpha_g \delta) = -\frac{\sqrt{\alpha_g \delta}}{2} \frac{\alpha_l}{\alpha_g} \frac{v_r^2}{|v_r|} \frac{1}{1 - \left(v_r^2/c_m^2\right)(1 + (v_r/|v_r|)\sqrt{\alpha_g \delta})^2}. \tag{5.70}$$

As illustrated by the numerical experiments in Section 7.1, the computation of such formal series is expected to be numerically stable for the pressure correction (2.8) but unstable for the pressure correction (2.9).

As soon as the series converges, computation of the approximations will be inexpensive:

- Matrix–vector products involving H are simple because at most only two or three elements of H are nonzero.
- The eigen-decomposition of $(B - \lambda(0)Id)$ is known explicitly and, consequently, (5.66) is easy to solve (in the space orthogonal to $x(0)$).
- the method quickly converges: in practice we restrict our study to $i = 1, 2$ (see the numerical experiments in Section 7.1 or 7.2).

As an illustration, in the case $v_r > 0$, for the pressure correction (2.8), we find the first-order perturbations of the eigenvalues (where we have set $\xi = v_r/c_m$):

$$\begin{aligned}
\lambda_{1_1} &= v_g - c_m + \epsilon \frac{1}{2} \frac{\alpha_l \rho_g}{\alpha_g \rho_l} c_m \frac{1}{(1 - \xi(1 + \sqrt{C_p}))(1 - \xi(1 - \sqrt{C_p}))} \\
\lambda_{2_1} &= v_l - \sqrt{C_p} v_r + \epsilon \frac{1}{2} \frac{\alpha_l \rho_g}{\alpha_g \rho_l} \frac{1}{\sqrt{C_p}} \frac{v_r}{1/(1 + \sqrt{C_p})^2 - \xi^2} \\
\lambda_{3_1} &= v_l + \sqrt{C_p} v_r + \epsilon \frac{1}{2} \frac{\alpha_l \rho_g}{\alpha_g \rho_l} \frac{1}{\sqrt{C_p}} \frac{-v_r}{1/(1 - \sqrt{C_p})^2 - \xi^2} \\
\lambda_{4_1} &= v_g + c_m + \epsilon \frac{1}{2} \frac{\alpha_l \rho_g}{\alpha_g \rho_l} c_m \frac{1}{(1 + \xi(1 + \sqrt{C_p}))(1 + \xi(1 - \sqrt{C_p}))}.
\end{aligned} \tag{5.71}$$

5.2. *Degenerate System*

It is well known that the derivation of stable approximations of the eigenvalues and eigenvectors is less straightforward and more CPU expensive when the unperturbed matrix has a Jordan block [2, 17, 23]. Without going into further detail, considering estimate (3.38), or expansion (4.58), we see that

$$\frac{\partial \lambda'_{2_i}}{\partial \epsilon} \simeq \frac{1}{\sqrt{\epsilon}} \tag{5.72}$$

and the derivation of stable algorithms to get the eigenstructure of such perturbation problems might be not obvious. As we want in this paper to use algorithms based on explicit expansion, we will only compute strictly diagonalizable models in the following numerical applications for simplicity.

Remark 5.4. Some authors suggest taking into account the ratio v_r/c_m in order to get approximations of the eigenvalues and vectors [18, 34, 35]. We think it would be difficult to extend the D.P.M. using the ratio v_r/c_m. Indeed, because of requirement (2.13), a direct computation shows that it would lead to a 2×2 Jordan block in the unperturbed matrix. Hence the derivation of stable approximate expressions might be less straightforward.

6. DERIVATION OF THE ROE SOLVER

Upwind schemes are widely used to approximate solutions of nonlinear hyperbolic systems [4, 9–11, 13, 15, 21, 30]. Application to two-phase flow is not straightforward but significant improvements have already been proposed by several authors [3, 18, 27, 29, 32–35]. Here, we do not discuss the additional difficulty due to the nonconservative form of the two-fluid model and, for simplicity, we do not deal with pressure corrections. We briefly explain how to build the Roe solver in one space dimension and refer the reader to [35] for a more comprehensive treatment.

The objective is to solve the partial differential system

$$\begin{aligned} &\partial_t U + \partial_x \Phi(U) = 0, \\ &U(0, x) = U_0(x), \\ &t > 0; \quad x \in \Omega \subset \mathcal{R}, \end{aligned} \tag{6.73}$$

where $U(x, t)$ is the phase space vector (3.19) and $U_0(x)$ is the initial data. Starting from a spatial mesh Ω,

$$\Omega = \bigcup_j [x_{j-1/2}, x_{j+1/2}], \tag{6.74}$$

and following [35], we define a local linearization $\breve{\alpha}_k$ of the nonconservative terms α_k at the interface $x_{j+1/2}$ which enables us to get a local conservative model. Hence, the physical flux Φ at the interface is given by

$$\Phi(U) = f(U) + p(U) \begin{pmatrix} 0 \\ 0 \\ \epsilon \breve{\alpha}_l \\ \breve{\alpha}_g \end{pmatrix}. \tag{6.75}$$

Integrating Eq. (6.73) on $[x_{j-1/2}, x_{j+1/2}] \times [t, t + dt]$ (Fig. 1), one can easily get the expression of the scheme

$$\frac{U_j^{t+dt} - U_j^t}{dt} + \frac{\Phi_{j+1/2} - \Phi_{j-1/2}}{dx} = 0 \tag{6.76}$$

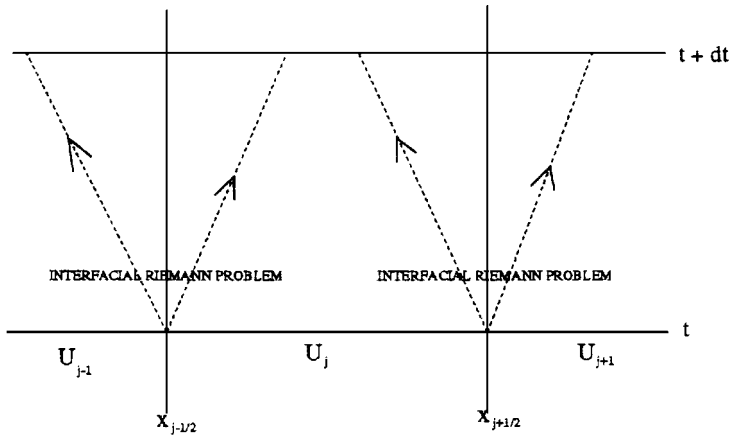

FIG. 1. Evolution in time with upwind schemes.

with

$$\Phi_{j+1/2} = \frac{1}{dt}\int_{[t,t+dt]} \Phi\big(U\big(x_{j+\frac{1}{2}}, s\big)\big)\, ds, \quad U_j^t = \frac{1}{dx}\int_{[x_{j-1/2}, x_{j+1/2}]} U(v,t)\, dv. \quad (6.77)$$

Hence, at each cell there is associated an average constant value U_j^t for $U(x,t)$ and, under suitable CFL condition to avoid interaction beetwen interfacial consecutive Riemann solutions, the numerical flux $\Phi_{j+1/2}$ is evaluated by the resolution of a Riemann problem at $x_{j+1/2}$ (Fig. 1). The way to approximate the solution of this Riemann problem leads to distinct schemes. We have used the Roe scheme [30] which consists of a local linearisation (6.78) of (6.73),

$$\partial_t U + \mathcal{A}_{\text{roe}}(U_j, U_{j+1})\, \partial_x U = 0, \quad (6.78)$$

where $\mathcal{A}_{\text{roe}}$ satisfies the properties,

$$\begin{aligned} &\mathcal{A}_{\text{roe}} \text{ diagonalizable} \\ &\mathcal{A}_{\text{roe}}(U,U) = \partial_U \Phi \\ &\mathcal{A}_{\text{roe}}(U_{j+1} - U_j) = \Phi(U_{j+1}) - \Phi(U_j). \end{aligned} \quad (6.79)$$

So, we obtain the expression of the numerical flux,

$$\Phi_{j+1/2} = \frac{\Phi(U_j) + \Phi(U_{j+1})}{2} - \frac{1}{2}|\mathcal{A}_{\text{roe}}|(U_{j+1} - U_j), \quad (6.80)$$

with the standard definitions:

$$\left|\begin{aligned} &\mathcal{A} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1}; \quad D = \text{diag}(\lambda_k), \\ &|D| = \text{diag}(|\lambda_k|) \\ &|\mathcal{A}| = \mathcal{P}|\mathcal{D}|\mathcal{P}^{-1}. \end{aligned}\right. \quad (6.81)$$

The computations yield the following Roe matrix at the cell interface $x_{j+1/2}$:

$$A_{roe}(U_j, U_{j+1}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\hat{v}_l^2 & 0 & 2\hat{v}_l & 0 \\ \check{\alpha}_g \partial_{U_1}\hat{p} & \check{\alpha}_g \partial_{U_2}\hat{p} - \hat{v}_g^2 & 0 & 2\hat{v}_g \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \check{\alpha}_l \partial_{U_1}\hat{p} & \check{\alpha}_l \partial_{U_2}\hat{p} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.82}$$

($\hat{}$ are the Roe average state quantities; see [35]). Hence, the perturbation methods described above are applied to get approximations of the eigenvalues and eigenvectors. We are now able to compute the Roe scheme for numerical experiments.

Remark 6.5. The shock-relation for the system is defined by

$$A_{roe}(U_j, U_{j+1})(U_{j+1} - U_j) = \Phi(U_{j+1}) - \Phi(U_j). \tag{6.83}$$

We want to point out that the change of variables (3.18) simply consists of a multiplication of the initial unknown vector by the diagonal matrix Q,

$$Q = \begin{pmatrix} \rho_l^o & 0 & 0 & 0 \\ 0 & \rho_g^o & 0 & 0 \\ 0 & 0 & \rho_l^o & 0 \\ 0 & 0 & 0 & \rho_g^o \end{pmatrix}, \tag{6.84}$$

and consequently does not modify the initial shock-relation which would be derived from the system (2.15).

7. COMPUTATIONAL RESULTS

In all the numerical applications, the gas is assumed to be perfect with $\gamma = 1.4$.

7.1. *Convergence Rate Estimate*

To test the convergence rate of the method, we have computed four Roe matrices for the two-fluid model; $A_{\text{roe}}^{\text{RTL}}$ and $A_{\text{roe}}^{\text{IT}}$ denote respectively the matrix obtained by the correction (2.8) and the correction (2.9) with $\epsilon = \frac{1}{20}$, whereas $\overline{A}_{\text{roe}}^{\text{RTL}}$ and $\overline{A}_{\text{roe}}^{\text{IT}}$ denote the ones obtained with $\epsilon = \frac{1}{500}$:

$$A_{\text{roe}}^{\text{RTL}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 578.9 & 575.9 & 1 & 0 \\ 1897.6 & 1887.3 & 0 & 2 \end{pmatrix}; \quad A_{\text{roe}}^{\text{IT}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 578.9 & 575.9 & 1 & 0 \\ 1892.7 & 1887.3 & 0 & 2 \end{pmatrix} \tag{7.85}$$

$$\overline{A}_{\text{roe}}^{\text{RTL}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 54 & 46.8 & 1 & 0 \\ 6739.6 & 5859.5 & 0 & 2 \end{pmatrix}; \quad \overline{A}_{\text{roe}}^{\text{IT}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 54 & 46.8 & 1 & 0 \\ 6552.2 & 5859.5 & 0 & 2 \end{pmatrix}. \tag{7.86}$$

We have computed a reference solution with a numerical computational routine and we

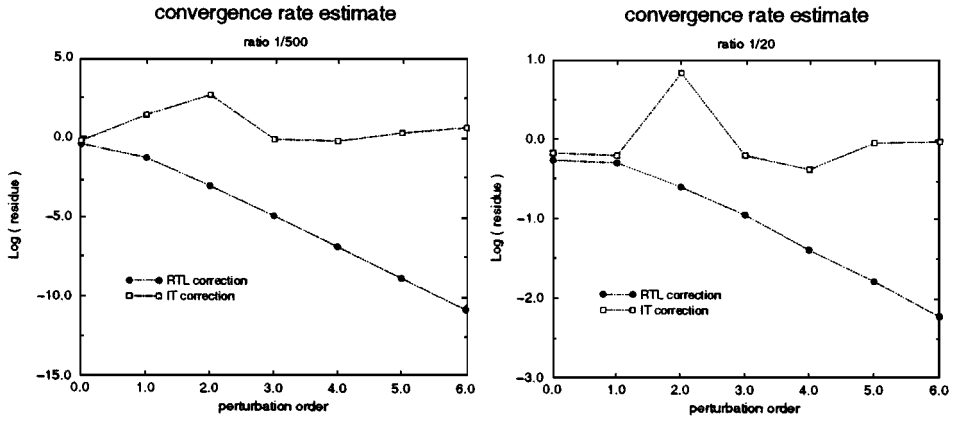

FIG. 2. Evolution of the residue r_n.

have defined the residue,

$$r_n = \frac{1}{16}\sum_{i,j=1}^{4}\left|\frac{|A^n_{\text{roe}}|(i,j) - |A_{\text{roe}}|(i,j)}{|A_{\text{roe}}|(i,j)}\right|, \tag{7.87}$$

where $|A^n_{\text{roe}}|$ is the absolute value of the Roe matrix obtained with the nth order approximation (5.65) and (5.66) of the eigenstructure and $|A_{\text{roe}}|$ the reference computational solution. We illustrate in Fig. 2 the evolution of the residue versus the perturbation order n for the several matrices mentioned above.

The numerical results are consistent with the mathematical results obtained in Subsection 5.1, in particular with the (5.67) and (5.69) condition numbers. Moreover, we see that the density perturbation method provides, for the correction (2.8), a quick algorithm which is of prime interest for industrial applications.

7.2. *Two-Phase Shock-Tube*

The shock-tube consists in a Riemann problem for the two-fluid model. In our study, the mathematical solution is composed of five constant states separated by shocks or rarefaction waves [20]. For two-fluid models, this is a test without a known solution for comparison but more interesting from a numerical stability point of view.

7.2.1. SHOCK1 (Influence of pressure corrections). The left and right states are defined in Table I. We have taken $\rho_l^o = 1000$ (Kg/m^3) and $\rho_g^o = 1$ (Kg/m^3). We have computed a first-order D.P.M. for the correction (2.8) and we have used a numerical routine for the correction (2.9). Figure 3 shows the evolution in time of the vapor velocity and of the void fraction. These results show that intermediate states are strongly dependent on the modeling

TABLE I

SHOCK 1	Left state	Right state
α_l	0.7	0.56
p(Pa)	96547	50000
v_g(m/s)	-0.4	-0.5
v_l(m/s)	0	0.2

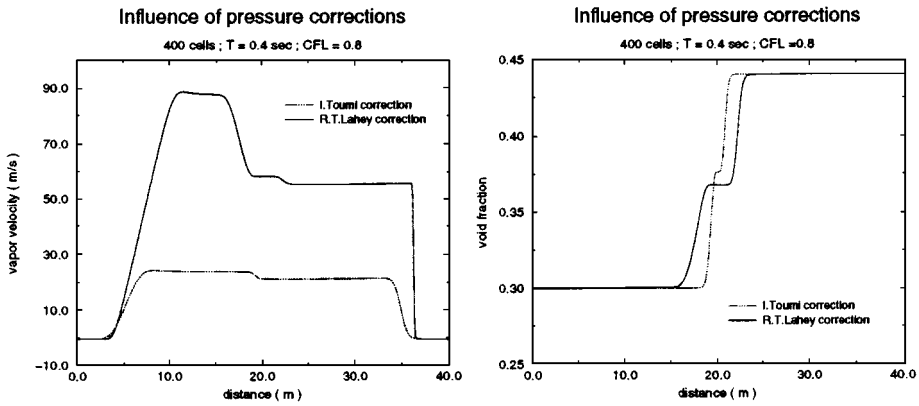

FIG. 3. Shock 1, influence of pressure corrections.

of the pressure correction chosen. We also want to underline that the use of a numerical routine for the correction (2.8) gives the same physical result as the use of the D.P.M., but it is quite a lot more expensive in CPU time.

7.2.2. SHOCK2 (A larger relative velocity shock). Using the ratio v_r/c_m, Tiselj and Petelin [34] have shown that the accuracy of the approximate expressions for the eigenvalues and eigenvectors is sufficient for $v_r < 0.05\, c_m$, but CPU expensive numerical procedures must be used otherwise. According to the mathematical result mentioned in Remark 5.4 and the numerical results of Section 7.1, we wanted to test the related limitation of the D.P.M. with the correction (2.8) (Table II). We have computed two values of $C_p = C^*$ defined in (2.8). Namely, in this vapor-velocity shock, $v_r \simeq c_m/2$, $\rho_g^0 = 20$ (Kg/m^3), and $\rho_l^0 = 1000$ (Kg/m^3). Figure 4 shows the evolution of the shock for the pressure and the ratio v_r/c_m; we see that for this shock the numerical results obtained are stable.

7.2.3. SHOCK3 (Influence of the Perturbation Order). In order to test the robustness of the method, we have computed a solution with a larger density ratio, $\rho_g^0 = 60$ (Kg/m^3) and $\rho_l^0 = 800$ (Kg/m^3), and with a void fraction close to 1. Initial shock is defined in Table III.

We see that the computed profile changes very much from the order 0 to the order 1 approximation, but we gain very little precision when a fourth-order expansion is used. We emphasize here that a fourth-order approximation can be easily computed, even if it seems to be rarely needed in practice (Fig. 5).

7.3. *Water Faucet Problem*

This numerical benchmark test was proposed by Ransom in [26]. No source terms are taken into account but Ransom has proposed an approximate analytical solution to the problem. Hence, this test is mentioned in several references.

TABLE II

SHOCK 2	Left state	Right state
α_l	0.71	0.7
p(Pa)	2.65×10^5	2.65×10^5
v_g(m/s)	65	50
v_l(m/s)	1	1

TABLE III

SHOCK 3	Left state	Right state
α_l	0.02	0.09
p(Pa)	$1.54\ 10^5$	$1.57\ 10^5$
v_g(m/s)	12	5
v_l(m/s)	4	1

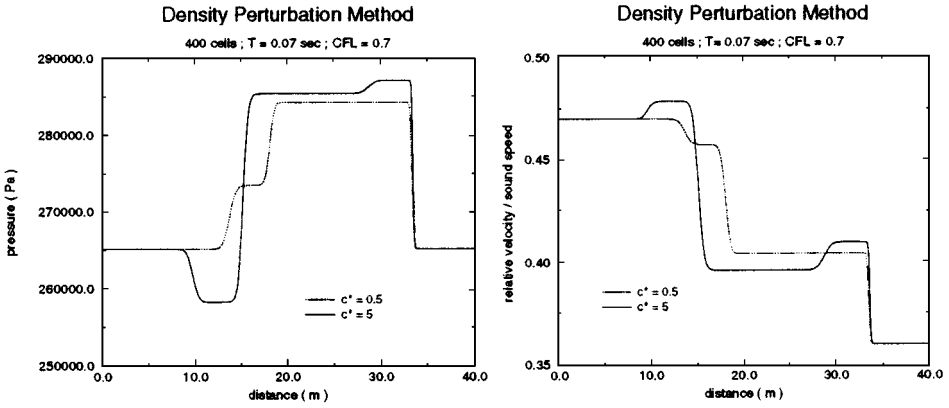

FIG. 4. Shock 2, a vapor-velocity shock.

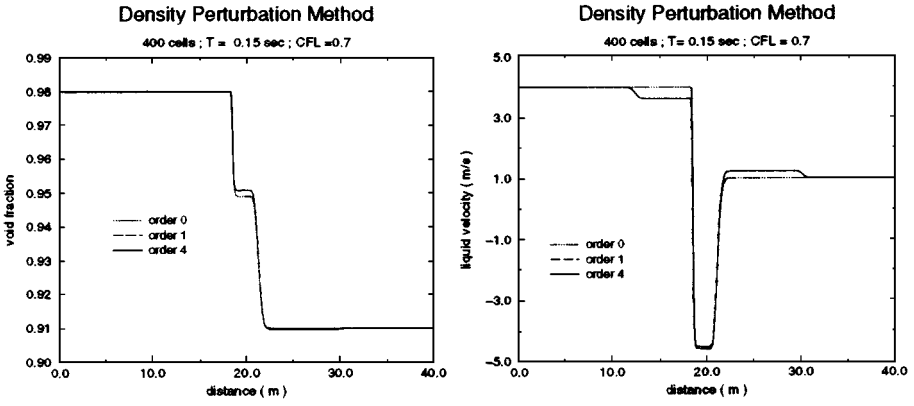

FIG. 5. Shock 3, influence of the perturbation order.

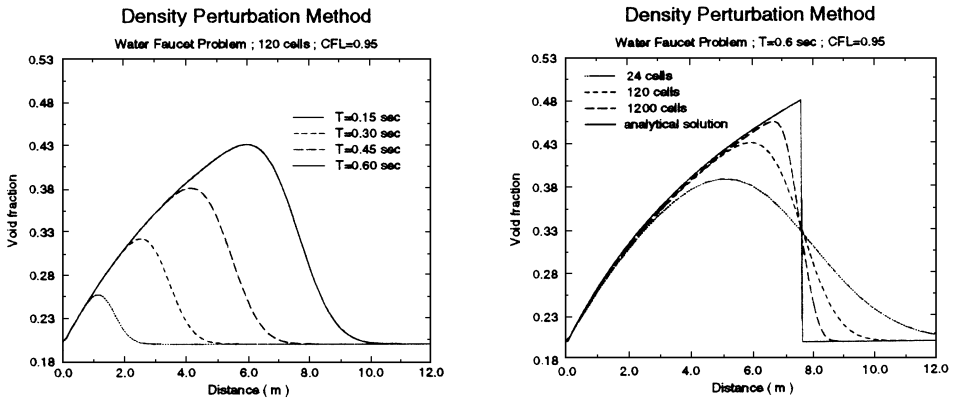

FIG. 6. Void fraction evolution.

We have computed an air–water density ratio $\epsilon = \frac{1}{1000}$ and a first-order D.P.M. for the correction (2.8). We study the action of gravity on a vertical water jet. At the initial state, a pipe is filled with a uniform columnn of water, the void fraction is 0.2, the velocity 10 m/s, and the pressure is 10^5 Pa. The boundary conditions are specified velocities of 10 m/s for the liquid and 0 m/s for the gas at the inlet, a constant pressure at the outlet. A void wave propagation is observed in the pipe (Fig. 6). In order to test the convergence and the stability character of the scheme, we have computed a spatial mesh refinement. There is no oscillation at the discontinuity of the void fraction even when the number of cells computed is high. Moreover, these results are in good agreement with the analytical solution proposed by Ransom (Fig. 6).

8. CONCLUSION

We have shown that the use of the density ratio provides a convenient way to study the convection part of several two-fluid models:

- Using perturbation analysis, the study of the hyperbolicity of such models can be done easily.
- We have shown that, for some pressure corrections, the unperturbed system obtained with this ratio does not contain a Jordan block. This is important since the perturbation of such matrices raises numerous numerical problems.
- Using a model proposed by Lahey [19], we have derived stable numerical results. A Roe scheme has been implemented and the eigenelements are approximated very economically.
- A large relative velocity-shock ($v_r \simeq c_m/2$) and the water faucet problem have been successfully tested.
- We also want to mention that a multidimensional extension of our work, on a model with interphase source terms, has been done by one of the authors in [5].

We think that the approach of the density perturbation method can be helpful in the understanding of phasic disequilibrium and can simplify multidimensional two-phase flow industrial computations.

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